LECTURE 29 CURVE SKETCHING TIPS, EXAMPLES AND INDETERMINATE FORMS

Questions about Intermediate Steps in Curve Sketching

Today, we go over some questions I collected from your emails for the past few days.

Example. Given the form of the derivative

$$f'(x) = (4\sin(x) - 4)\left(\sqrt{2}\cos(x) + 1\right),$$

discuss the behaviour of the function f on $[0, 2\pi]$.

Example. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x-1)^2 (x-2)(x-4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

Indeterminate Forms

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, 0^0 and 1^∞ are all indeterminate forms. It relates to the rate at which 0, 1 and ∞ is approached, and if one cannot clarify such rates, these expressions don't have meanings.

Example. Here, we give a few examples that lead to the above indeterminate forms.

(1) $\lim_{x\to 0} \frac{\sin(x)}{x} = $ "$\frac{0}{0}$".
(2) $\lim_{x\to\infty} \frac{\ln(x)}{2\sqrt{x}} = $ "$\frac{\infty}{\infty}$".
(3) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = $ "$\infty \cdot 0$".
(4) $\lim_{x\to 0} \left(\frac{1}{\sin(x)} - \frac{1}{x}\right) = $ "$\infty - \infty$".
(5) $\lim_{x\to 0} x^x = $ "0^0".
(6) $\lim_{x\to 0} \left(1 + \frac{1}{x}\right)^x = $ "1^∞".

L'Hôpital's Rule enables us to compute the above limits. It is important to understand the hypothesis of the rule.

Theorem. *(L'Hôpital's Rule) Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then,*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Remark 1. This applies to the indeterminante form "$\frac{0}{0}$" since $f(a) = g(a) = 0$. We will see that with some manipulations, other forms can be derived from this form alone.

Remark 2. Sometimes, we need to do L'Hôpital's Rule repeatedly.